The Lee Fields Medal IV

1. What is the last digit of 7^{2022}?

2. Describe a way of generating better and better approximations to $2^{\sqrt{2}}$ using surds.

3. Take an odd number $n \geq 3$ and square it. Then divide it by two and look at the two integers closest to what you get, say b and c.

   Show that no matter what odd number n you choose, the numbers (n, b, c) are the side lengths of a right-angled-triangle.

   For example, let $n = 3$ so that $n^2 = 9$. This number divided by two is 4.5, with nearest integers $b = 4$ and $c = 5$, and there is a right-angled triangle with side-lengths 3, 4, 5.

4. Find the equation of the parabola(s) $y = ax^2 + bx + c$ which contains the points $P(1,0)$, $Q(2,3)$, $R(3,10)$.

5. Four equilateral triangles are arranged around a square which has area 12. What's the total <u>shaded</u> area?
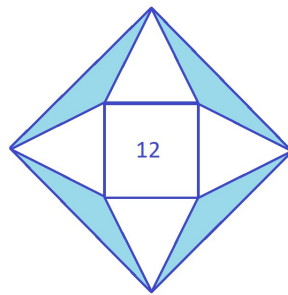
6. With the aid of a diagram, explain why for $x > 0$:

$$\tan^{-1}(x) + \tan^{-1}\left(\frac{1}{x}\right) = 90°.$$

7. Six cups and saucers come in pairs: there are two cups and saucers which are red, two white, and two black. If the cups are randomly placed onto the saucers (one each), find the probability that no cup is on a saucer of the same colour.

8. Consider three points $(1, 2.1), (2, 3.8), (3, 6.1)$ on the plane. Find the line $y = mx$ that minimises the sum of the *squared deviations* of the points from the line.
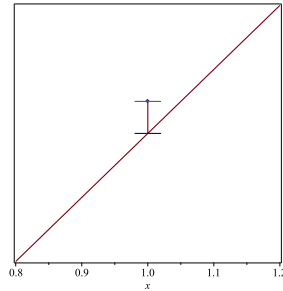
Zooming in near a point (x_i, y_i), we see its deviation from the line $y = mx$. This deviation is the vertical distance from the point to the line, and the *squared deviation* of (x_i, y_i) from $y = mx$ is equal to $(y_i - mx_i)^2$. In the case of the point $(1, 2.1)$, this squared deviation is $(m(1) - 2.1)^2 = (m - 2.1)^2$.

9. Johnny & Mary live in a town with only eight pubs but they still want to do the 12 Pubs of Christmas. They decide to start in Pub A and after each drink move to one of the three adjacent pubs whether they have been there before or not. If they do *not* have to visit every pub, can Johnny & Mary end up back in Pub A for their 12th drink? Justify your answer.
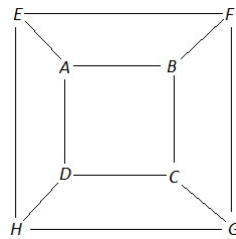
10. Two cubes are used in a calendar to display the day for the current month as shown below. List what should be written on the sides of each cube so that all days 1-31 can be displayed by the calendar: